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Rational lift of the combinatorial R-matrix

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Abstract. We present a rational lift of the combinatorial R-matrix for tensor products of rectangular tableaux. The key ingredients are a connection between the product of rectangular tableaux (in the plactic monoid) and multiplication of certain "tableau matrices," and a birational map from these matrices to the Grassmannian.

Résumé. Nous présentons un relèvement rationnel de la matrice R combinatoire pour les produits tensoriels de tableaux rectangulaires. Les ingrédients principaux sont, d'abord, un lien entre les produits de tels tableaux (dans le monoïde plaxique) et la multiplication de certaines "matrices de tableaux," ainsi qu'une application birationnelle de ces matrices dans la grassmannienne.

Keywords: crystal, geometric crystal, Kirillov-Reshetikhin module, R-matrix, tropicalization

1 Introduction

The term R-matrix refers to an isomorphism $V \otimes W \to W \otimes V$, where V and W are representations of a quantum group. In this paper, we are interested in a family of finite-dimensional modules for the quantum affine algebra of type $A_{n-1}^{(1)}$, known as Kirillov-Reshetikhin modules. Each of these modules has a corresponding crystal, and for any pair of crystals B_1 and B_2, there is a unique crystal isomorphism $\widetilde{R} : B_1 \otimes B_2 \to B_2 \otimes B_1$, called the combinatorial R-matrix. The Kirillov-Reshetikhin crystals in affine type A are indexed by rectangular partitions with at most $n-1$ rows, and they are modeled by the set of semistandard Young tableaux of the given rectangular shape; thus, we view the combinatorial R-matrix as a map on pairs of tableaux. For a combinatorial description of this map, see Section 2.2 below.

The theory of geometric crystals seeks to find geometric analogues of crystals, that is, algebraic varieties equipped with rational functions which "behave similarly" to the operations on crystals. In the cases of interest, the rational functions can be tropicalized to obtain piecewise-linear functions which describe the combinatorial crystal operations. Berenstein and Kazhdan ([1]) defined a geometric analogue of the tensor product of crystals, and there is a natural notion of morphism of geometric crystals; thus, the geometric R-matrix ought to be an isomorphism $R : X_1 \times X_2 \to X_2 \times X_1$, where X_1, X_2 are affine geometric crystals corresponding to Kirillov-Reshetikhin modules. In the case of single-row tableaux, there is a well-known affine geometric crystal, and a compatible geometric R-matrix ([11]). Interestingly, this geometric R-matrix is the unique (non-trivial) solution to a certain matrix equation, which we now describe.

Given an n-tuple $x = (x_1, \ldots, x_n)$ of nonzero complex numbers, let $M(x)$ be the $n \times n$ matrix with x_i in position (i,i), 1 in position $(i+1,i)$ for $i = 1, \ldots, n-1$, and an indeterminate λ in position $(1,n)$. For example,

$$M(x_1, x_2, x_3) = \begin{pmatrix} x_1 & 0 & \lambda \\ 1 & x_2 & 0 \\ 0 & 1 & x_3 \end{pmatrix}.$$

Theorem 1.1 ([11], [5])**.** *Fix $x, y \in (\mathbb{C}^\times)^n$. For generic x and y, the matrix equation $M(x)M(y) = M(y')M(x')$ has two solutions: the trivial solution $y' = x, x' = y$, and a non-trivial solution given by*

$$x_i' = x_i \frac{\kappa_i}{\kappa_{i+1}}, \qquad y_i' = y_i \frac{\kappa_{i+1}}{\kappa_i} \qquad \text{where} \qquad \kappa_i = \sum_{j=0}^{n-1} y_i \cdots y_{i+j-1} x_{i+j+1} \cdots x_{i+n-1}$$

and subscripts are interpreted modulo n.

The authors of [8] showed that the tropicalization of the map $(x, y) \mapsto (y', x')$ gives a piecewise-linear formula for the combinatorial R-matrix for a tensor product of one-row crystals.

In [3], we constructed an affine geometric crystal on $\mathrm{Gr}(k,n) \times \mathbb{C}^\times$ which corresponds to the Kirillov-Reshetikhin crystals with $n-k$ rows (see also [7] for a different construction). In this paper, we exploit the structure of $\mathrm{Gr}(k,n) \times \mathbb{C}^\times$ to obtain a geometric R-matrix for tensor products of Kirillov-Reshetikhin crystals with $n-k$ rows. Just as in [11], our geometric R-matrix will be a solution to a matrix equation.

There are various applications of the geometric and combinatorial R-matrices. The connection between matrix multiplication and one-row tableaux is used in [8] to construct a rational lift of the RSK correspondence. The single-row geometric R-matrix in Theorem 1.1 appeared independently in Lam and Pylyavskyy's study of the totally positive part of the loop group ([5]), where they used it to prove a factorization theorem (the matrix $M(x)$ is a shifted version of their "whirl"). They also studied the invariants of this map, which they call loop symmetric functions; these share some of the nice properties of symmetric functions ([4]).

The combinatorial R-matrix can be used to define a cellular automaton called the box-ball system. This automaton exhibits soliton behavior, and Lam-Pylyavskyy-Sakamoto ([6]) have a conjectural formula for the soliton lengths (and "higher order" information about the solitons) in terms of the tropicalization of certain loop symmetric functions.

They proved the first case of their conjecture using the single-row geometric R-matrix. We are optimistic that the general geometric R-matrix in this paper can be used to settle the remaining cases.

The paper is arranged as follows. In Section 2, we review the combinatorial maps and objects that appear in this paper. In Section 3, we show how to associate a "tableau matrix" to a rectangular tableau, and we explain the precise sense in which the product of these matrices tropicalizes to the product of the corresponding tableaux (Theorem 3.9). In Section 4, we define a map from tableau matrices to the Grassmannian, and then a map from the Grassmannian to the loop group; the effect is to "enhance" the tableau matrices so that the analogue of Theorem 1.1 has a unique solution. In Section 5, we define the geometric R-matrix, and state some of its key properties. Finally, in Section 6, we derive the single-row geometric R-matrix of [11],[8],[5] from the $\mathrm{Gr}(n-1,n)$ version of our construction, in the case $n = 3$. Proofs of the results in Sections 3-5 will appear in [3], [2].

2 Preliminaries

Throughout this paper, we fix an integer $n \geq 2$. We write $[i,j]$ to denote the (possibly empty) set $\{x \in \mathbb{Z} : i \leq x \leq j\}$, and we write $[i]$ to denote $[1,i] = \{1,\ldots,i\}$.

2.1 Tableaux and Gelfand-Tsetlin patterns

Definition 2.1. *A* semistandard Young tableaux *(SSYT) is a filling of a Young diagram with numbers in $[n]$, so that the rows are weakly increasing, and the columns are strictly increasing. The partition corresponding to the Young diagram is the* shape *of the tableau.*

A Gelfand-Tsetlin pattern *(GT pattern) is a triangular array of non-negative integers $(A_{ij})_{1\leq i\leq j\leq n}$ satisfying the inequalities*

$$A_{i,j+1} \geq A_{ij} \geq A_{i+1,j+1}$$

for $1 \leq i \leq j \leq n-1$.

There is a simple bijection between Gelfand-Tsetlin patterns and SSYTs. Given a Gelfand-Tsetlin pattern (A_{ij}), the associated tableau T is described as follows: the number of j's in the ith row of T is $A_{ij} - A_{i,j-1}$ (we use the convention that $A_{i,i-1} = 0$).

In this paper, we will focus primarily on *rectangular* SSYTs, that is, tableaux consisting of k rows of some common length L. We will typically fix the number of rows $k \leq n-1$, and allow the row length L to vary.

Definition 2.2. *A* k-row rectangular GT pattern *is an array of $k(n-k)+1$ nonnegative integers*

$$(B_{ij}, L)_{1\leq i\leq k, i\leq j\leq i+n-k-1}$$

such that the triangular array defined by

$$A_{ij} = \begin{cases} B_{ij} & \text{if } i \leq k \text{ and } i \leq j \leq i+n-k-1 \\ L & \text{if } i \leq k \text{ and } i+n-k \leq j \leq n \\ 0 & \text{if } i > k \end{cases}$$

is a Gelfand-Tsetlin pattern.

The bijection between Gelfand-Tsetlin patterns and tableaux restricts to a bijection between k-row rectangular GT patterns and k-row rectangular SSYTs, as illustrated by the following example.

Example 2.3. *Suppose $n = 5$ and $k = 3$. Here is an example of a 3-row rectangular GT pattern (B_{ij}, L), the "full" GT pattern (A_{ij}) that it represents, and the corresponding SSYT.*

$$\begin{array}{ccccc} & & 2 & & \\ & 4 & & 2 & \\ 6 & & 3 & & 1 \\ & & & 1 & \end{array} \quad \longleftrightarrow \quad \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 4 & & 2 & & & \\ & & 6 & & 3 & & 1 & & \\ & 6 & & 6 & & 1 & & 0 & \\ 6 & & 6 & & 6 & & 0 & & 0 \end{array} \quad \longleftrightarrow$$

1	1	2	2	3	3
2	2	3	4	4	4
3	5	5	5	5	5

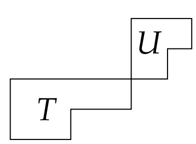
The 6 in the bottom left corner is L, the length of each row. The "full" GT pattern in the middle is obtained by filling in the bottom left corner with L, and the bottom right corner with 0.

Definition 2.4. *For $k \in [n-1]$ and $L \geq 1$, let $B^{k,L}$ denote the set of semistandard tableaux of shape (L^k) with entries in $[n]$.*

Using the above bijection, we will identify $B^{k,L}$ with the set of k-row rectangular Gelfand-Tsetlin patterns with fixed length parameter L, so that an element of $B^{k,L}$ is represented by the $k(n-k)$ integer coordinates B_{ij}.

2.2 Tableau product, combinatorial R-matrix, and Schützenberger involution

Definition 2.5. *Let T and U be semistandard tableaux. Define the* tableau product *of T and U, denoted $T * U$, to be the rectification of the skew shape obtained by placing U to the northeast of T, as shown here:*



This product is associative, but it is not commutative. However, when both of the tableaux have rectangular shape, there is a well-defined way to "swap the order" of the tableaux.

Theorem/Definition 2.6 ([9]). *Suppose $(T, U) \in B^{k_1,L_1} \times B^{k_2,L_2}$. There exists a unique pair $(U', T') \in B^{k_2,L_2} \times B^{k_1,L_1}$ such that $T * U = U' * T'$. The map $\widetilde{R} : (T, U) \mapsto (U', T')$ is called the* combinatorial R-matrix.

Remark 2.7. *As mentioned in the introduction, the combinatorial R-matrix is the unique crystal isomorphism of the affine crystals $B^{k_1,L_1} \otimes B^{k_2,L_2}$ and $B^{k_2,L_2} \otimes B^{k_1,L_1}$.*

We come to the central problem of this paper.

Problem 2.8. *Find a formula for the Gelfand-Tsetlin coordinates of U' and T' in terms of the Gelfand-Tsetlin coordinates of T and U.*

Our solution to this problem makes use of the Schützenberger involution $\widetilde{S}$, a well-known operation on semistandard tableaux ([10]). In the case of rectangular tableaux, this operation has a very simple description.

Lemma 2.9. *The Schützenberger involution acts on a rectangular SSYT by rotating the tableau 180 degrees, and replacing each entry i with $n + 1 - i$. It acts on the k-row rectangular GT pattern (B_{ij}, L) by keeping L fixed, and replacing B_{ij} with $L - B_{k-i+1,n-j}$.*

3 Lifting the tableau product

3.1 Tropicalization

Tropicalization is the procedure which transforms a subtraction-free rational function into a piecewise-linear function by replacing multiplication with addition, division with subtraction, and addition with the operation min. The tropicalization of a constant is zero.[1] If $g : \mathbb{C}^n \to \mathbb{C}^m$ is a subtraction-free rational map, then we denote by $\mathrm{Trop}(g)$ its tropicalization, i.e., the piecewise-linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ obtained by tropicalizing each component function of g.

Example 3.1. *Suppose* $g = \dfrac{x_1^2 x_2 + x_3 + 1}{x_2^5 + 2x_1 x_3}$. *Then*

$$Trop(g) = \min(2x_1 + x_2, x_3, 0) - \min(5x_2, 0 + x_1 + x_3).$$

3.2 Rational Gelfand-Tsetlin patterns and tableau matrices

In this section, instead of viewing the entries of a Gelfand-Tsetlin pattern as integers satisfying certain inequalities, we will view them as non-zero complex numbers.

[1] For a more careful definition of tropicalization, see e.g. [1].

Definition 3.2. *Fix $k \leq n-1$. Set $D_k = \{(i,j) \in \mathbb{Z}^2 \,|\, 1 \leq i \leq k \text{ and } i \leq j \leq i+n-k-1\}$, and define $\mathbb{T}_k = (\mathbb{C}^\times)^{D_k} \times \mathbb{C}^\times$. A point $X = (X_{ij}, t) \in \mathbb{T}_k$ will be called a k-*row rational Gelfand-Tsetlin pattern*.*

Note that the index set for $\mathbb{T}_k$ in Definition 3.2 is the same as the index set for k-row rectangular GT patterns in Definition 2.2.

To work with rational GT patterns, it is convenient to map them to a matrix, as we now describe.

Definition 3.3. *Let E_{ij} denote the $n \times n$ matrix with a 1 in position (i,j), and zeros elsewhere. For $1 \leq i \leq n-1$ and $z \in \mathbb{C}^\times$, define*

$$x_{-i}(z) = zE_{ii} + z^{-1}E_{i+1,i+1} + E_{i+1,i} + \sum_{j \neq i,i+1} E_{jj}.$$

For $1 \leq i \leq n$ and $z \in \mathbb{C}$, define

$$t_i(z) = zE_{ii} + \sum_{j \neq i} E_{jj}.$$

For example, if $n = 3$, then we have

$$x_{-2}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 1 & z^{-1} \end{pmatrix} \qquad \text{and} \qquad t_3(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix}.$$

Definition 3.4.

(a) For $i \leq j$, define

$$M_{[i,j]}(z_i, \ldots, z_j) = x_{-i}(z_i)x_{-(i+1)}(z_{i+1}) \cdots x_{-(j-1)}(z_{j-1})t_j(z_j).$$

(b) Define $\Phi_k : \mathbb{T}_k \to GL_n$ by

$$\Phi_k(X_{ij}, t) = \prod_{i=k}^{1} M_{[i,i+n-k]}(X_{ii}, X_{i,i+1}, \ldots, X_{i,i+n-k-1}, t)$$

where the terms of the product are arranged from left to right in decreasing order of i.

We will call $\Phi_k(X_{ij}, t)$ a tableau matrix.

Example 3.5. *Suppose $n = 4$ and $k = 2$, and $X = (X_{11}, X_{12}, X_{22}, X_{23}, t) \in \mathbb{T}_2$. Define $x_{ij} = X_{ij}/X_{i,j-1}$, where we set $X_{i,i+k} = t$ and $X_{i,i-1} = 1$ for each i. Then we have*

$$\Phi_2(X) = M_{[2,4]}(X_{22}, X_{23}, t)M_{[1,3]}(X_{11}, X_{12}, t)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_{22} & 0 & 0 \\ 0 & 1 & x_{23} & 0 \\ 0 & 0 & 1 & x_{24} \end{pmatrix} \begin{pmatrix} x_{11} & 0 & 0 & 0 \\ 1 & x_{12} & 0 & 0 \\ 0 & 1 & x_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & 0 & 0 & 0 \\ x_{22} & x_{12}x_{22} & 0 & 0 \\ 1 & x_{12}+x_{23} & x_{13}x_{23} & 0 \\ 0 & 1 & x_{13} & x_{24} \end{pmatrix}.$$

Remark 3.6. *The variables $x_{ij} = X_{ij}/X_{i,j-1}$ are the "de-tropicalized version" of the quantities $b_{ij} = B_{ij} - B_{i,j-1}$, which give the number of j's in the ith row of the SSYT corresponding to the rectangular GT pattern (B_{ij}, L).*

3.3 Matrix multiplication and the tableau product

Suppose (B_{ij}, L) and (B'_{ij}, L') are two rectangular GT patterns, corresponding to rectangular tableaux T and U. The entries in the Gelfand-Tsetlin pattern for the product $T * U$ are piecewise-linear functions in the coordinates B_{ij}, L, B'_{ij}, L'. It turns out that these piecewise-linear functions are the tropicalizations of rational functions associated with the product of two tableau matrices. We now give a precise formulation of this statement. This part is rather technical, and for the rest of the paper it is only necessary to keep in mind the following slogan:

The product of tableau matrices tropicalizes to the product of rectangular tableaux.

Definition 3.7. *Let $S = \{s_1 < \cdots < s_m\}$ be a subset of $[n]$. For convenience, set $s_0 = 0$. For $i \in [s_m]$, define $\alpha_S(i) = s_r$, where $i \in [s_{r-1}+1, s_r]$. Define*

$$D_S = \{(i,j) \mid 1 \le i \le s_m \text{ and } i \le j \le i + n - \alpha_S(i) - 1\}$$

and set $\overline{D_S} = D_S \cup S$. Denote an element of $(\mathbb{C}^\times)^{\overline{D_S}}$ by (Z_{ij}, Z_s), where $(Z_{ij}) \in (\mathbb{C}^\times)^{D_S}$, and $(Z_s) \in (\mathbb{C}^\times)^S$. Define a map $\Phi_S : (\mathbb{C}^\times)^{\overline{D_S}} \to GL_n$ by

$$(Z_{ij}, Z_s) \mapsto \prod_{i=s_m}^{1} M_{[i,i+n-\alpha_S(i)]}(Z_{ii}, Z_{i,i+1}, \ldots, Z_{i,i+n-\alpha_S(i)-1}, Z_{\alpha_S(i)})$$

where the terms of the product are arranged from left to right in decreasing order of i.

Definition 3.8. *For $\ell, k \in [n-1]$, define*

$$S_{\ell,k} = [\max(1, \ell+k-n), \min(\ell,k)] \cup [\max(\ell,k), \min(n, \ell+k)].$$

Theorem 3.9 ([2])**.** *Suppose $X \in \mathbb{T}_\ell$ and $Y \in \mathbb{T}_k$, and set $S = S_{\ell,k} = \{s_1 < \cdots < s_m\}$.*

(a) For each $(i,j) \in D_S$ (respectively, each $s \in S$), there is a uniquely defined subtraction-free rational function $Z_{ij}(X,Y)$ (respectively, $Z_s(X,Y)$) such that

$$\Phi_\ell(X)\Phi_k(Y) = \Phi_S(Z_{ij}(X,Y), Z_s(X,Y)).$$

(b) Let $B = (B_{ij}, L)$ be an ℓ-row rectangular GT pattern, and let $B' = (B'_{ij}, L')$ be a k-row rectangular GT pattern. For $1 \le i \le j \le n$, define

$$C_{ij} = \begin{cases} Trop(Z_{ij})(B,B') & \text{if } (i,j) \in D_S \\ Trop(Z_{\alpha_S(i)})(B,B') & \text{if } i \le s_m \text{ and } (i,j) \notin D_S \\ 0 & \text{if } i > s_m. \end{cases}$$

Then (C_{ij}) is a Gelfand-Tsetlin pattern, and it corresponds to the semistandard tableau $T * U$, where T corresponds to B and U corresponds to B'.

This theorem shows that solutions to the matrix equation $\Phi_k(X)\Phi_\ell(Y) = \Phi_\ell(Y')\Phi_k(X')$ provide candidates for the geometric R-matrix. The problem is that there are many possible solutions. In the remainder of this paper, we explain how to add additional constraints, and thereby pick out a single solution.

4 The Grassmannian and the loop group

4.1 From rational GT patterns to the Grassmannian

Definition 4.1. *Define the map $\pi^k : GL_n \to Gr(k,n)$ to be the "projection" of an invertible matrix onto the k-dimensional subspace spanned by its first k columns. Define the map*

$$\Theta_k : \mathbb{T}_{n-k} \to Gr(k,n) \times \mathbb{C}^\times$$

by $\Theta_k(X_{ij}, t) = (M, t)$, where $M = (\pi^k \circ \Phi_{n-k})(X_{ij}, t)$.

Definition 4.2. *Let M be a matrix representative for a point in the Grassmannian $Gr(k,n)$. We use the convention that M is an $n \times k$ matrix. For each subset $J \subset [n]$ of size k, let $P_J = P_J(M)$ denote the maximal minor of M using the rows in J. The P_J are the* Plücker coordinates *of M.*

Proposition 4.3 ([3])**.** *The map Θ_k is an open embedding of $\mathbb{T}_{n-k}$ into $Gr(k,n) \times \mathbb{C}^\times$. The (rational) inverse is given by $\Theta_k^{-1}(M,t) = (X_{ij}, t)$, where*

$$X_{ij} = \frac{P_{[i,j] \cup [n-k+j-i+2,n]}(M)}{P_{[i+1,j] \cup [n-k+j-i+1,n]}(M)}$$

for $1 \le i \le n-k$ and $i \le j \le i+k-1$.

We call Θ_k the *Gelfand-Tsetlin parameterization* of $\mathrm{Gr}(k,n) \times \mathbb{C}^\times$. Using this parameterization, we may transfer information obtained from the geometry of the Grassmannian to information about rational GT patterns, which in turn can be tropicalized to obtain information about tableaux.

4.2 From the Grassmannian to the loop group

Definition 4.4. *Let λ be an indeterminate, and let $\widehat{GL_n} := GL_n(\mathbb{C}(\lambda))$ denote the group of invertible $n \times n$ matrices with entries in the field of rational functions in λ. Call this group the* loop group. *Given $A \in \widehat{GL_n}$ and $a \in \mathbb{C}$, we will write $A|_{\lambda=a}$ to denote the matrix obtained by evaluating λ at a. Also, we will write $\pi_a^k(A)$ to denote the point of $Gr(k,n)$ spanned by the first k columns of $A|_{\lambda=a}$. If a is a root of some entry of A or the first k columns of $A|_{\lambda=a}$ are linearly dependent, then $\pi_a^k(A)$ is undefined.*

Definition 4.5. *Define a rational map $g_k : Gr(k,n) \times \mathbb{C}^\times \to \widehat{GL_n}$ by $g_k(M,t) = A$, where A is defined by*

$$A_{ij} = c_{ij} \frac{P_{[j-k+1,j-1]\cup\{i\}}(M)}{P_{[j-k,j-1]}(M)}, \qquad c_{ij} = \begin{cases} 1 & \text{if } j \le k \\ t & \text{if } j > k \text{ and } i \ge j \\ \lambda & \text{if } j > k \text{ and } i < j. \end{cases}$$

The intervals in this formula are considered to lie in $\mathbb{Z}/n\mathbb{Z}$ (so $[n-1,3] = \{n-1,n,1,2,3\}$).

For example, if $(M,t) \in \mathrm{Gr}(2,4) \times \mathbb{C}^\times$, then

$$g_2(M,t) = \begin{pmatrix} \frac{P_{14}}{P_{34}} & 0 & \lambda & \lambda\frac{P_{13}}{P_{23}} \\ \frac{P_{24}}{P_{34}} & \frac{P_{12}}{P_{14}} & 0 & \lambda \\ 1 & \frac{P_{13}}{P_{14}} & t\frac{P_{23}}{P_{12}} & 0 \\ 0 & 1 & t\frac{P_{24}}{P_{12}} & t\frac{P_{34}}{P_{23}} \end{pmatrix}.$$

The map g_k has the following important properties.

Proposition 4.6 ([3], [2])**.** *Let $(M,t) = \Theta_k(X_{ij},t)$, and let $A = g_k(M,t)$. Then we have the following:*

(a) $A|_{\lambda=0} = \Phi_k(X_{ij},t)$.

(b) $\pi^k(A) = M$. *(Note that the first k columns of A do not depend on λ.)*

(c) The matrix $A|_{\lambda=(-1)^{k-1}t}$ has rank k.

(d) Suppose $(N,s) \in Gr(\ell,n) \times \mathbb{C}^\times$, with $s \neq (-1)^{k+\ell}t$. Then $\pi^k_{(-1)^{k-1}t}(A \cdot g_\ell(N,s)) = M$.

Part (a) says that the matrix A is a "deformation" of the matrix $\Phi_k(X_{ij},t)$. The usefulness of this deformation is that it allows the subspace M (and thus the rational GT pattern (X_{ij},t)) to be recovered, even after multiplying on the right by $g_\ell(N,s)$. We will exploit this property of A in the next section.

5 The geometric R-matrix

Let $\mathbb{X}_r = \mathrm{Gr}(r,n) \times \mathbb{C}^\times$. We will define a rational map $R : \mathbb{X}_\ell \times \mathbb{X}_k \to \mathbb{X}_k \times \mathbb{X}_\ell$ which, when composed with the Gelfand-Tsetlin parameterizations of $\mathbb{X}_\ell$ and $\mathbb{X}_k$, tropicalizes to the combinatorial R-matrix. Before presenting the definition, we give some motivation for it.

Theorem 3.9 states, roughly speaking, that if the rational GT patterns $X = (X_{ij}, s), Y = (Y_{ij}, t)$ correspond to SSYTs T, U, then the matrix $\Phi_\ell(X)\Phi_k(Y)$ "tropicalizes to" the SSYT $T * U$. Thus, it is natural to look for rational GT patterns $Y' = (Y'_{ij}, t), X' = (X'_{ij}, s)$ such that $\Phi_k(Y')\Phi_\ell(X') = \Phi_\ell(X)\Phi_k(Y)$. Proposition 4.6 suggests that we should ask for even more, namely, that

$$g_\ell(\Theta_\ell(X))g_k(\Theta_k(Y)) = g_k(\Theta_k(Y'))g_\ell(\Theta_\ell(X')). \tag{5.1}$$

Suppose such Y' and X' exist, and let C denote the matrix in (5.1). By Proposition 4.6(d) (and properties of the Schützenberger involution $\widetilde{S}$), we would have

$$\Theta_k(Y') = (\pi^k_{(-1)^{k-1}t}(C), t) \qquad \text{and} \qquad \Theta_\ell(X') = ((S \circ \pi^\ell_{(-1)^{\ell-1}s} \circ \text{flip})(C), s).$$

Here flip is reflection over the antidiagonal (i.e., $A_{ij} \mapsto A_{n+1-j,n+1-i}$), and S is the "geometric Schützenberger involution." (The map S acts on $(M, s) \in \mathbb{X}_\ell$ by sending (M, s) to $(Z_{ij}, s) = \Theta_\ell^{-1}(M, s)$, replacing Z_{ij} with $s/Z_{n-\ell-i+1,n-j}$, and then applying Θ_ℓ. See [3],[2] for more information.) With this in mind, we define the geometric R-matrix.

Definition 5.1. *Suppose $x = (N, s) \in \mathbb{X}_\ell$ and $y = (M, t) \in \mathbb{X}_k$. Let $C = g_\ell(x)g_k(y)$. The* geometric R-matrix *is the map $R : \mathbb{X}_\ell \times \mathbb{X}_k \to \mathbb{X}_k \times \mathbb{X}_\ell$ given by $R(x, y) = (y', x')$, where*

$$y' = (\pi^k_{(-1)^{k-1}t}(C), t) \qquad \text{and} \qquad x' = ((S \circ \pi^\ell_{(-1)^{\ell-1}s} \circ \text{flip})(C), s).$$

Theorem 5.2 ([2])**.** *Suppose $x = (N, s) \in \mathbb{X}_\ell$ and $y = (M, t) \in \mathbb{X}_k$, and let $(y', x') = R(x, y)$. Then we have*

$$g_k(y')g_\ell(x') = g_\ell(x)g_k(y).$$

Although this result is to be expected from the preceding discussion, its proof requires heavy computation.

The next result shows that the geometric R-matrix provides a solution to Problem 2.8.

Theorem 5.3 ([2])**.** *Define a map $\widehat{R} : \mathbb{T}_{n-\ell} \times \mathbb{T}_{n-k} \to \mathbb{T}_{n-k} \times \mathbb{T}_{n-\ell}$ by*

$$\widehat{R}(X_{ij}, s, Y_{ij}, t) = ((\Theta_k^{-1} \times \Theta_\ell^{-1}) \circ R \circ (\Theta_\ell \times \Theta_k))(X_{ij}, s, Y_{ij}, t). \tag{5.2}$$

(a) The map $\widehat{R}$ is given by subtraction-free rational functions in the variables X_{ij}, s, Y_{ij}, t.

(b) If $B = (B_{ij}, L_1)$ is an $(n-\ell)$-row rectangular GT pattern and $B' = (B'_{ij}, L_2)$ is an $(n-k)$-row rectangular GT pattern, then Trop$(\widehat{R})(B, B') = \widetilde{R}(B, B')$.

We conclude by stating some additional properties of the geometric R-matrix.

Theorem 5.4 ([2])**.**

(a) *R commutes with the affine geometric crystal operators defined in [3].*

(b) *R is an involution.*

(c) *Consider the triple product $\mathbb{X}_{k_1} \times \mathbb{X}_{k_2} \times \mathbb{X}_{k_3}$. Let $R_{i,i+1}$ denote the map which acts as the geometric R-matrix on factors i and $i+1$, and the identity on the other factor. We have the Yang-Baxter relation*

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

6 Example

Consider the case $n = 3, \ell = k = 2$. Suppose $X = (X_{11}, X_{12}, s), Y = (Y_{11}, Y_{12}, t) \in \mathbb{T}_1$, and set $(Y', X') = \widehat{R}(X, Y)$ ($\widehat{R}$ is defined in (5.2)). Define

$$x_1 = X_{11} \qquad x_2 = X_{12}/X_{11} \qquad x_3 = s/X_{12}$$

and define y_i, x'_i, y'_i similarly (using t instead of s for y_i and y'_i). Note that $s = x_1x_2x_3$ and $t = y_1y_2y_3$. Let $(M, t) = \Theta_2(Y)$, and let $A = g_2(\Theta_2(X))$. By definition, $Y' = \Theta_2^{-1}(M', t)$, where

$$M' = A|_{\lambda=-t} \cdot M = \begin{pmatrix} x_1 & 0 & -y_1y_2y_3 \\ 1 & x_2 & 0 \\ 0 & 1 & x_3 \end{pmatrix} \begin{pmatrix} y_1 & 0 \\ 1 & y_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1y_1 & -y_1y_2y_3 \\ y_1 + x_2 & x_2y_2 \\ 1 & y_2 + x_3 \end{pmatrix}.$$

We compute

$$\begin{aligned} P_{12}(M') &= y_1y_2(x_1x_2 + y_3x_2 + y_3y_1) = y_1y_2\kappa_3 \\ P_{13}(M') &= y_1(x_3x_1 + y_2x_1 + y_2y_3) = y_1\kappa_2 \\ P_{23}(M') &= x_2x_3 + y_1x_3 + y_1y_2 = \kappa_1 \end{aligned}$$

where $\kappa_1, \kappa_2, \kappa_3$ are the polynomials defined in Theorem 1.1. By Proposition 4.3 and the definition of y'_i, we have

$$y'_1 = \frac{P_{13}(M')}{P_{23}(M')} = y_1\frac{\kappa_2}{\kappa_1} \qquad y'_2 = \frac{P_{12}(M')}{P_{13}(M')} = y_2\frac{\kappa_3}{\kappa_2} \qquad y'_3 = t\frac{P_{23}(M')}{P_{12}(M')} = y_3\frac{\kappa_1}{\kappa_3}.$$

Similarly, one computes $x'_i = x_i\dfrac{\kappa_i}{\kappa_{i+1}}$, so $\widehat{R}$ agrees with the map from Theorem 1.1 in this case.

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